TAMARI LATTICES AND NONCROSSING
PARTITIONS IN TYPE B AND BEYOND

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ABSTRACT. The usual, or type A_n, Tamari lattice is a partial order on T_n^A, the triangulations of an $(n+3)$-gon. We define a partial order on T_n^B, the set of centrally symmetric triangulations of a $(2n + 2)$-gon. We show that it is a lattice, and that it shares certain other nice properties of the A_n Tamari lattice, and therefore that it deserves to be considered the B_n Tamari lattice.

We define a bijection between T_n^B and the noncrossing partitions of type B_n defined by Reiner. For S any subset of $[n]$, Reiner defined a pseudo-type BD_n^S, to which is associated a subset of the noncrossing partitions of type B_n. We show that the elements of T_n^B which correspond to the noncrossing partitions of type BD_n^S posess a lattice structure induced from their inclusion in T_n^B.

INTRODUCTION

The usual (or type A_n) Tamari lattice is a partial order on T_n^A, the set of triangulations of an $(n + 3)$-gon. By a triangulation of a polygon, we mean a division of the polygon into triangles by connecting pairs of its vertices with straight lines which do not cross in the interior of the polygon. The purpose of this paper is to define and investigate the properties of an analogous lattice defined on centrally symmetric triangulations of a $2n + 2$-gon, which we call the type B_n Tamari lattice. More explanation for why these lattices should be associated to the reflection groups of types A_n and B_n will be given below.

We begin by reviewing some features of the type A_n Tamari lattice. Conventionally, we will number the vertices of our $(n + 3)$-gon clockwise from 0 to $n + 2$, with a long top edge connecting vertices 0 and $n + 2$. An example triangulation is shown in Figure 1 below.

Let $S \in T_n^A$. As in [Lee], we colour the chords of S red and green, as follows. A chord C of S is the diagonal of a quadrilateral $Q(C)$ in S. If C is the diagonal of $Q(C)$ which is connected to the vertex with the largest label, we colour it green; otherwise we colour it red. In Figure 1, the red chords are indicated by thick lines.
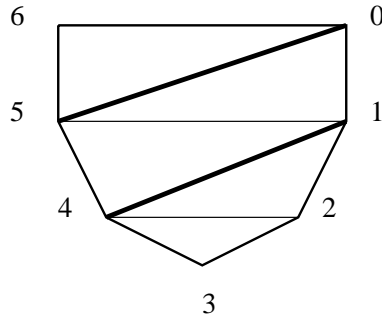
Figure 1

We partially order T_n^A by giving covering relations: T covers S if they coincide except that some green chord in S has been replaced by the other diagonal of $Q(C)$ (which is red). This is one way to construct the the Tamari lattice, which was introduced in [Tam] and which has since been studied by several authors (see [HT, Pal, Mar, BW2]).

Although this is not clear from the elementary description given here, the Tamari lattice should be thought of as belonging to type A. One indication of why can be found in [BW2], where it is shown that T_n^A is a quotient of the weak order on the symmetric group S_{n+1} (the type A_n reflection group). Another reason is that the elements of T_n^A index clusters in the A_n root system (see Fomin and Zelevinsky [FZ]). Once one has the idea that the Tamari lattice belongs to type A, it is natural to ask whether there exist Tamari lattices in other types.

For reasons which we shall go into further below, the B_n triangulations, denoted T_n^B, are the triangulations of a centrally symmetric $(2n+2)$-gon which are themselves centrally symmetric (i.e. fixed under a half-turn rotation). These triangulations have already appeared in the work of Simion [Sim], and in [FZ] where they index the clusters in the B_n root system. One goal of our paper is to define a partial order on T_n^B and to prove that it is a lattice. The definition is analogous to that already given for the A_n Tamari lattice: it is given in terms of covering relations, and S covers T in T_n^B if S is obtained from T by replacing a symmetric pair of chords $C, \bar{C}$ by the other diagonals of $Q(C), Q(\bar{C})$. The details of the definition are a trifle complicated, so we defer them for the main body of the paper. This definition was arrived at independently and more or less simultaneously by Reading [Rea]. He has also proved that T_n^B is a lattice, using a rather different approach. Two alternative partial orders on T_n^B with similar (but somewhat easier to describe) covering relations were suggested by Simion [Sim]; one is studied further in [Sa1]. Since neither of these is a lattice, neither is completely satisfying as a type B analogue of the usual Tamari lattice.

An important property of T_n^A is that there is a natural (though not order-preserving) bijection from T_n^A to the (classical) noncrossing partitions, NC_n^A. The type B_n version of noncrossing partitions, NC_n^B was introduced by Reiner [Rei]. We show that there is a bijection from T_n^B to NC_n^B similar to that from T_n^A to NC_n^A.

In the same paper where Reiner defined the noncrossing partitions of type B_n, he defined a more general notion of noncrossing partitions of (pseudo-)type BD_n^S, for $S \subset [n]$. The noncrossing partitions of type BD_n^S are a subset of NC_n^B. Using the above bijection between T_n^B and NC_n^B, for any S, we describe the subset of

T_n^B which corresponds to the noncrossing partitions of type BD_n^S, and we show that the order induced on this subset of T_n^B by its inclusion in T_n^B gives it a lattice structure, which we call the Tamari lattice of pseudo-type BD_n^S.

We show that T_n^B (and the Tamari lattices of type BD_n^S also) have an unrefinable chain of left modular elements, a property also shared by the usual Tamari lattice [BS]. One consequence of this, due to Liu [Liu], is that these lattices have EL-labellings. Using these labellings, we show that, as for the usual Tamari lattice (see [BW2]), the order complex of any interval is either homotopic to a sphere or contractible. (This result on homotopy types of order complexes of intervals was also obtained by Reading [Rea].)

From the results in this paper one could proceed in two directions. One direction is to consider the existence of Tamari lattices in all Coxeter types. The other direction is to investigate further the lattices defined here, to see how many more of the properties of the usual Tamari lattice carry over.

Type B Triangulations

Recall that the B_n Weyl group consists of signed permutations of n. We can think of these as permutations of $\{1, \ldots, n, \bar{1}, \ldots, \bar{n}\}$ fixed under interchanging i and $\bar{i}$ for all $1 \le i \le n$. By analogy, B_n triangulations, T_n^B, are defined to be type A triangulations of a $(2n + 2)$-gon fixed under a half-turn. There is general consensus that this is the correct choice of B_n triangulation: see [Sim], [FZ].

We number the vertices of our standard $(2n + 2)$-gon clockwise from 1 to $n + 1$ and then from $\bar{1}$ to $\overline{n+1}$. A typical triangulation is shown in Figure 2.

We will frequently distinguish two types of chords: *pure* and *mixed*. A chord is pure if it connects two barred vertices or two unbarred vertices; otherwise it is mixed. For $S \in T_n^B$, consider a chord C of S. The chord C is the diagonal of a quadrilateral, which we denote $Q(C)$. If C is pure, then we colour it red if $Q(C)$ contains another vertex of the same type as those of C whose label is higher, and green otherwise. If it is mixed, we colour it red if $Q(C)$ contains an unbarred vertex whose label is higher than the label of the unbarred vertex of C, or a barred vertex whose label is higher than the label of the barred vertex of C. Otherwise we colour it green. In Figure 2, the red chords are indicated by thick lines.
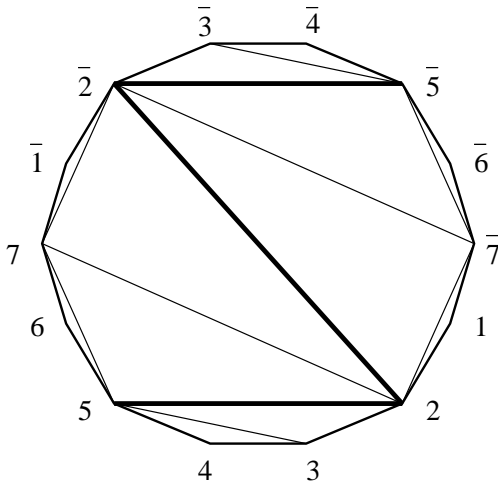
Figure 2

For C a chord, we write $\bar{C}$ for its symmetric partner (that is to say, the image of C under a half turn). Observe that C and $\bar{C}$ are assigned the same colour.

Lemma 1. *Consider a chord C in a triangulation S. Let S' be the triangulation obtained by replacing C by C', the other diagonal of $Q(C)$, and also replacing $\bar{C}$ by $\bar{C}'$. Then the colours of C in S and C' in S' are opposite.*

Proof. The proof is just a case-by-case check of the possible configurations of the four vertices of $Q(C)$: all of one type, three of one type and one of the other, or two of each type. $\square$

We can now state the first main theorem of this paper (which, as was already remarked, was arrived at and proved independently and more or less simultaneously by Reading [Rea]).

Theorem 1. *There is a lattice structure on T_n^B whose covering relations are given by $S \lessdot T$ iff S and T differ in that green chords $C, \bar{C}$ in S are replaced in T by the other diagonals of $Q(C)$ and $Q(\bar{C})$ (which will be red). Note that we allow $C = \bar{C}$ (i.e. C being a diameter). We call this lattice the B_n Tamari lattice.*

Proof. The proof of this theorem will take the rest of this section and all of the next. We begin with a quick outline. We will associate to every triangulation $S \in T_n^B$ a *bracket vector* $r(S)$ which is an n-tuple of elements from $[0, n-1] \cup \{\infty\}$. We will define a partial order on T_n^B in terms of bracket vectors, and then show that its covering relations are as given in the statement of the theorem, and that it is a lattice.

The first ingredient in our proof of Theorem 1 is some further analysis of the red and green chords of triangulations. Fix a triangulation S. For $1 \le i \le n$, look for a vertex of the polygon which is connected to i, starting at $\bar{1}$ and searching clockwise. If none is found before reaching the vertex next counterclockwise from i (i.e. the vertex $i - 1$, unless $i = 1$, in which case the vertex $\overline{n+1}$), then $C_i(S)$ is the edge segment connecting i and the next vertex counterclockwise. Otherwise, if a vertex connected to i was found, then $C_i(S)$ is the chord of S connecting i to that vertex. Let $R(S)$ be the set of the $C_i(S)$ which are chords rather than edge segments, together with their symmetric partners.

Lemma 2. *For any triangulation S, $R(S)$ consists of the red chords; the chords not in $R(S)$ are green.*

Proof. Pick a chord in $R(S)$. Since the colouring is symmetric, we may assume that the chord is $C_i(S)$ for some i. It follows that $Q(C_i(S))$ contains a vertex greater than i, and hence that $C_i(S)$ is red.

Now consider a chord C not in $R(S)$. Suppose first that C is pure; we may assume that it connects i and j with $i > j$. Since $C \ne C_i(S)$, $C_i(S)$ divides $Q(C)$ from all those vertices with unbarred labels greater than i, so C is green. Next suppose that C is mixed, connecting i and $\bar{j}$. Now $C_i(S)$ divides $Q(C)$ from those vertices with unbarred labels greater than i, and $\overline{C_j(S)}$ divides it from those vertices with barred labels greater than $\bar{j}$. So C is green. $\square$

Lemma 3. *Let M be a region of the $(2n+2)$-gon, that is to say, the convex hull of some subset of the vertices of the $(2n+2)$-gon. Then there is a unique way to triangulate M using only green chords.*

Proof. The way to do it is as follows: connect every unbarred vertex to the largest unbarred vertex, every barred vertex to the largest barred vertex, and, if both exist, connect the largest barred and largest unbarred vertices. It is easy to see that all these chords are green.

Uniqueness is clear in the case where there is one type of vertex (that is to say, barred or unbarred) which appears at most once in the region. So suppose we are not in this case, and fix a triangulation of M using green chords. We wish to show that it is the triangulation defined in the previous paragraph.

Write x for the largest unbarred vertex in M. Observe that there must be at least one mixed chord in the triangulation, but no $i < x$ can have a mixed chord inside M attached to it, because this would imply by Lemma 2 that there was a red chord inside M, contradicting our assumption. Thus there is a mixed chord in the interior of M which is connected to x, say $x\bar{j}$.

Now consider the triangle containing $x\bar{j}$, which is on the side of $x\bar{j}$ with the smaller unbarred labels and the larger barred labels. Suppose first that its third vertex is barred. In this case, $x\bar{j}$ is not green, contradicting our assumption. So the third vertex must be unbarred, say z. Now $z\bar{j}$ cannot be green. Thus, it must be a boundary of M. This implies that $\bar{j}$ must be the greatest barred vertex of M. Thus, we have shown that our triangulation of M contains the chord connecting the greatest barred vertex and the greatest unbarred vertex. This chord divides M into two subregions which fall into the simple type (no more than one barred vertex or no more than one unbarred vertex) for which uniqueness is clear. This establishes that the triangulation with which we began must coincide with that described in the first paragraph of this proof. $\square$

The type A analogue of Lemma 3 was proved in [Lee].

Lemma 4. *For any triangulation S, S is the unique triangulation whose set of red chords is exactly $R(S)$.*

Proof. Let T be a triangulation whose set of red chords is $R(S)$. The chords of $R(S)$ divide the $(2n+2)$-gon up into regions which are triangulated by green chords of T, but by Lemma 3 there is a unique way to do this, which must be that of S. So T coincides with S. $\square$

Bracket Vectors in types A and B

We briefly recall some well-known facts about the type A Tamari lattice, which serve as motivation for our work in type B.

Any triangulation $S \in T_n^A$ has a bracket vector $r(S) = (r_1(S), \ldots, r_{n+1}(S))$. Let $v_i(S)$ be the least vertex attached to i. Then $r_i(S) = i - 1 - v_i(S)$. For example, the bracket vector of the triangulation shown in Figure 1 is (0,0,0,2,4). This approach to representing elements of the Tamari lattice goes back to [HT], though we make some different choices of convention here.

Proposition 1 [HT]. *An $(n+1)$-tuple of positive integers is a bracket vector for some triangulation in T_n^A iff it satisfies the following two properties:*

(i) For $1 \le i < j \le n+1$, $r_i \le r_j - (j-i)$ provided $r_j - (j-i)$ is non-negative.
(ii) $0 \le r_i \le i-1$. $\square$

The order relation on triangulations has a simple interpretation in terms of bracket vectors, which we summarize in the following proposition:

Proposition 2 [HT, Mar]. *The lattice structure on T_n^A can be described as follows:*
(i) $S \le T$ iff $r_i(S) \le r_i(T)$ for all i.
(ii) $r_i(S \wedge T) = \min(r_i(S), r_i(T))$.
(iii) For x any $n+1$-tuple of numbers satisfying only the second condition of Proposition 1, there is a unique triangulation $\uparrow(x)$ such that for $S \in T_n^A$,

$$r_i(S) \ge x_i \text{ for all } i \text{ iff } S \ge \uparrow(x).$$

(iv) $r(S \vee T) = \uparrow(\max(r(S), r(T)))$, where max is taken coordinatewise. $\square$

We now proceed to describe a similar construction in type B. To a triangulation $S \in T_n^B$ we associate a bracket vector $r(S) = (r_1(S), \ldots, r_n(S))$, as follows. For $1 \le i \le n$, let $v_i(S)$ denote the end other than i of $C_i(S)$. If the counter-clockwise distance from $i-1$ to $v_i(S)$ is less than or equal to $n-1$, set $r_i(S)$ to be that distance. Otherwise, set $r_i(S) = \infty$. Thus, the triangulation shown in Figure 2 has bracket vector $(0, \infty, 0, 0, 2, 0)$.

Conventions regarding ∞. ∞ *is considered to be greater than any integer. ∞ plus an integer (or ∞) equals ∞.*

Lemma 5. *From the bracket vector $r(S)$, $C_i(S)$ can be determined as follows:*
(i) If $r_i(S) = 0$ then $C_i(S)$ is the edge segment extending counter-clockwise from i.
(ii) If $0 < r_i(S) < n$ then $C_i(S)$ connects i to the vertex $r_i(S)+1$ vertices counter-clockwise from i.
(iii) If $r_i(S) = \infty$ then $C_i(S) = i\bar{j}$ where j is the least vertex satisfying $r_j(S) - j \ge n - i$.

Proof. The cases $r_i(S) = 0$ and $0 < r_i(S) < n$ are obvious.

Suppose $r_i(S) = \infty$. In this case, certainly $C_i(S) = i\bar{j}$ for some j. It follows that $j\bar{i}$ is also a chord of S. Thus $r_j(S) - j \ge n - i$. Now, for $m < j$, it cannot be that vertex m is connected to $\bar{i}$, since then $C_i(S)$ would be $i\bar{m}$, not $i\bar{j}$. Also, it cannot be that m is connected to any vertex less than $\bar{i}$, since then $C_m(S)$ would divide j from $\bar{i}$. Thus, $r_m(S) - m < n - i$. It follows that j is the least number satisfying $r_j - j \ge n - i$, as desired. $\square$

Corollary. *The map from T_n^B to bracket vectors is injective.*

Proof. Given $r(S)$, we can determine $C_i(S)$ for all i. Their union together with their symmetric partners gives $R(S)$, and, by Lemma 4, determines S. $\square$

Proposition 3. B_n *bracket vectors are n-tuples of symbols from $[0, n-1] \cup \{\infty\}$ characterized by the following two properties:*
(i) For $1 \le i < j \le n$, $r_i \le r_j - (j-i)$ if $r_j - (j-i)$ is non-negative.
(ii) If $\infty > r_i \ge i$, then $r_{n+i-r_i} = \infty$.

Proof. Clearly a bracket vector satisfies condition (i). Suppose that $\infty > r_i \geq i$. Then i is connected to $\overline{n+i-r_i}$. By symmetry, $n+i-r_i$ is connected to $\bar{i}$, and therefore $r_{n+i-r_i} = \infty$. Thus a bracket vector also satisfies condition (ii).

Let r be a vector satisfying (i) and (ii). Recall that in Lemma 5, we showed that r determines C_i for all i. By conditions (i) and (ii), the C_i determined by r do not cross each other. Let R be the union of the C_i together with their symmetric partners. The chords of R divide the $(2n+2)$-gon into regions. Construct a triangulation S by triangulating the regions with green chords, using the construction of Lemma 3. Now, for each i, $C_i(S)$, if it is not an edge segment, is a red chord of S, and is therefore contained in R. Among the chords connected to i and contained in R, C_i is the first when encountered proceeding clockwise from $\bar{1}$, and therefore $C_i(S) = C_i$, and so $r(S) = r$, as desired. $\square$

We will now define an order on T_n^B. For $S, T \in T_n^B$, let $S \leq T$ iff for all i, $r_i(S) \leq r_i(T)$.

Proposition 4. *The covering relations in this order on T_n^B are exactly those described by Theorem 1.*

Proof. We begin by proving some lemmas.

Lemma 6. *Let $S, T \in T_n^B$ such that $S \lessdot T$. Then there exists some k such that $r_j(S) = r_j(T)$ for all $j \neq k$. Further:*

If $C_k(S) = ka$ is pure, then $C_k(T)$ connects k to the endpoint of $C_a(S)$ which is not a.

If $C_k(S) = k\bar{a}$ is mixed, then $C_k(T) = k\bar{b}$ where b is the largest number $a > b > k$ such that $r_b(S) = \infty$, or $b = k$ if there is no such number.

Conversely, if S and T are two triangulations with $r_i(S) = r_i(T)$ for $i \neq k$ for some k, and $C_k(S)$ and $C_k(T)$ are related as decribed above, then $S \lessdot T$ in T_n^B.

Proof. Note that our assumption is that $S \lessdot T$ with respect to the order defined just above in terms of bracket vectors — this lemma is part of the proof that the covering relations of this partial order are as described in the statement of Theorem 1. We write s_i for $r_i(S)$ and t_i for $r_i(T)$.

First, we prove the forward direction. Throughout this paragraph, (i) and (ii) refer to conditions (i) and (ii) of Proposition 3, which describe when an n-tuple is a legal B_n bracket vector. Suppose that s_i and t_i coincide for $i > k$, but $t_k > s_k$. We divide into cases. We suppose first that $C_k(S) = ka$ is pure. Then $t_k \geq s_k + s_a + 1$ by applying (i) at (k, a). Now $d = (s_1, \ldots, s_{k-1}, s_k + s_a + 1, s_{k+1}, \ldots, s_n)$ is a valid bracket vector: it satisfies (i) at (k, j) with $k < j$ because t does, while it satisfies (i) at (j, k) with $a < j < k$ because s does, and it satisfies (i) at (j, k) with $j < a$ because s satisfies (i) at (j, a), and it clearly satisfies (ii). It is clear that $s < d \leq t$, so $t = d$. The description of $C_k(T)$ in the statement of the lemma follows immediately.

Now suppose that $C_k(S) = k\bar{a}$ is mixed. Observe first that $a > k$ because $r_k(S) < \infty$. Let $\bar{b}$ be the first vertex encountered counterclockwise proceeding from $\bar{a}$ such that $r_b(S) = \infty$. Set $x = n + k - b$. If no such vertex is encountered before reaching $\bar{k}$, set $b = k$ and $x = \infty$. Then let $d = (s_1, \ldots, s_{k-1}, x, \ldots, s_n)$. This is a valid bracket vector. Since $t_i = s_i$ for $i > k$, $t_k \geq x$. Thus $t \geq d > s$, so $t = d$, as desired. Again, the statement in the lemma describing $C_k(T)$ follows immediately.

Finally, we prove the converse. Given such S and T, we know that $r_i(S) = r_i(T)$ for $i \neq k$, and $r_k(S) < r_k(T)$. We remark that there is no legal bracket vector lying betwen $r(S)$ and $r(T)$, and we are done. $\square$

Lemma 7. *Let $S \lessdot T$ in T_n^B. The k be as in the statement of Lemma 6, so that $r_i(S) = r_i(T)$ for $i \neq k$. Then:*

If $C_k(S)$ is pure, then $C_i(S) = C_i(T)$ for $i \neq k$.

If $C_k(S)$ is mixed, then $C_i(S) = C_i(T)$ for $i \neq k, i \neq b$, where $k\bar{b} = C_k(T)$.

Proof. As in the previous proof, we write s_i for $r_i(S)$ and t_i for $r_i(T)$.

Consider first the case where $C_k(S)$ is pure. Clearly, if $s_i = t_i \neq \infty$, then $C_i(S) = C_i(T)$. So suppose that $s_i = t_i = \infty$. In this case, recall from Lemma 5 that $C_i(S) = i\bar{x}$, where x is the smallest vertex with $s_x - x \geq n - i$. Similarly, let $C_i(T) = i\bar{y}$, so y is the smallest vertex with $t_y - y \geq n - i$. We could only have that $x \neq y$ if $y = k$ and $x \neq k$. Now $t_k - k = s_k + s_a + 1 - k = t_a - (k - (s_a + 1)) = t_a - a$. Since $a < k$, k cannot be the smallest vertex y with $t_y - y \geq n - i$, so $y = k$ is impossible.

Now consider the case where $C_k(S)$ is mixed. Let $C_k(T) = k\bar{b}$. Let i be neither k not b. As before, the only problematic case is when $s_i = t_i = \infty$, and $C_i(T) = i\bar{k} \neq C_i(S)$. If $i \geq a$ (and $C_i(T) = i\bar{k}$) then $C_i(S) = i\bar{k}$ also. If $a < i < b$, then $s_i = \infty$ contradicts the characterization of b given in Lemma 6. If $i < b$, then, since $C_k(T) = k\bar{b}$, k cannot be connected to i in T, contradicting our assumption. This completes the proof of the lemma. $\square$

Lemma 8. *Let $S \lessdot T$ in T_n^B, and let $r_j(S) = r_j(T)$ for all $j \neq k$. Then the only red chords of T which are not red chords of S are $C_k(T)$ and its symmetric partner, and all the red chords of S are also chords of T (though not necessarily red).*

Proof. By Lemma 7, if $C_k(S)$ is pure, then it is immediate that the only red chords of T which are not red chords of S are $C_k(T)$ and its symmetric partner. If $C_k(S)$ is mixed, then if we write $C_k(T) = k\bar{b}$, it is clear that the only red chords of T which are not red in S are $C_k(T)$ and $C_b(T)$ and their symmetric partners — but in fact $C_b(T)$ is the symmetric partner of $C_k(T)$, and the first statement of the lemma is proved.

To prove the second statement, write A for $C_k(S)$, and suppose first that $A = ka$ is pure.

Let $C_k(T)$ be kv, where v may be barred or unbarred. Any red chord of S other than A is $C_i(S)$ for some $i \neq k$, and is therefore, by Lemma 7, also a red chord of T. We must now dispose of the chord A, if it is not an edge segment. If A is a red chord of T, we are done, so suppose otherwise. Consider the division of T into regions by its red chords. We know that kv and av are red chords of T, so if there were a red chord of T crossing A, it would have to be iv for some $a < i < k$. But the fact that $s_i = t_i$ would then force $C_i(S)$ to cross A also, which is impossible, since they are both chords of S.

Thus, a and k are in the same region of T. Since k is the largest unbarred vertex in this region, k and a are connected by a green chord of T, by the construction of Lemma 3. So A is a green chord in T.

Suppose now that $C_k(S) = k\bar{a}$ is mixed. We continue to denote $C_k(S)$ by A. Let $C_k(T) = k\bar{b}$. Let $D = C_b(S)$.

As before, any red chord of S other than A or D is also a red chord of T, so we need only worry about A and D. Suppose A is not a red chord of T. Then, as in

the case where A is pure, we check that in order for a red chord of T to cross A, the corresponding red chord of S would also cross A, which is impossible. Since A connects the largest unbarred vertex and the largest barred vertex of the region of T containing it, by the construction of Lemma 3, it is a green chord of T.

Now consider D. Note that D is mixed, since $s_b = t_b = \infty$. The argument now proceeds in the same way as for A. □

We now begin the proof of Proposition 4 proper. Let $S \lessdot T$ in the order on T_n^B defined by $S \leq T$ iff $r_i(S) \leq r_i(T)$ for all i. We wish to show that S and T are related by a diagonal flip as in the statement of Theorem 1.

By Lemma 8, we can consider the division of S and T into regions by the red chords of S. In any of these regions, the chords of S are those of the unique triangulation of the region by green chords. Since all the red chords of T are red chords of S, except for $C_k(T)$ and its symmetric partner, the same thing is true for T, except in the region(s) which contain $C_k(T)$ and $\overline{C_k(T)}$. Thus S and T coincide except in the region containing $C_k(T)$ which we denote Z, and the region containing $\overline{C_k(T)}$, which we denote $\bar{Z}$. Z and $\bar{Z}$ may coincide.

Consider the boundary of the region Z. Consider first the case where $A = ka$ is pure. Let $C_k(T) = kv$ where v may represent a barred or unbarred vertex. Begin at k. Proceeding counter-clockwise around the boundary of Z, the next vertex is a, and the next is v. In S, all of these vertices are connected to the largest unbarred vertex of Z, say i. (Note that $i \neq k$ since $C_k(S) = A$.) In T, all the unbarred vertices of Z are connected to i except a; k and v are connected by a red chord. Thus, we see that S and T differ in that the green diagonal of $ikav$ has been replaced by the red diagonal of $ikav$, and similarly for the symmetric partner of $ikav$.

Next consider the case where $A = k\bar{a}$ is mixed. Consider the boundary of Z, beginning at k and proceeding counter-clockwise. The next vertex encountered is $\bar{a}$. So $\bar{a}$ is the largest barred vertex of Z, so (by the construction of Lemma 3) all the barred vertices of Z are connected to $\bar{a}$ in S. It follows that none of the barred vertices of Z except the smallest and the largest can have the corresponding entries of $r(S)$ be ∞. Thus, writing $C_k(T) = k\bar{b}$, we have that $\bar{b}$ is the smallest barred vertex of Z. Thus, S and T differ in Z within the quadrilateral defined by the largest and smallest barred and unbarred vertices of Z; in S the larger ones are connected, while in T the two smaller ones are conected. If $Z \neq \bar{Z}$, then the same analysis holds in $\bar{Z}$.

Thus, we have shown that if $S \lessdot T$ in T_n^B, then they are related by a minimal flip as in Theorem 1. We must now check that if S and T are related by a minimal flip as in Theorem 1, then $S \lessdot T$ in T_n^B.

So suppose that S and T are related by a minimal flip as in Theorem 1: that is to say, there is a chord C of S which is green, such that T can be obtained from S by replacing C and $\bar{C}$ by the other diagonals of $Q(C)$ and $Q(\bar{C})$. It is a case-by-case check, based on the positions of the four vertices of $Q(C)$, that $r_i(S) = r_i(T)$ for $i \neq k$ for some k. One then checks that $C_k(S)$ and $C_k(T)$ are related as in Lemma 6. By the converse direction of Lemma 6, this then implies that $S \lessdot T$ in T_n^B. This completes the proof of Proposition 4. □

Our next goal is to prove that the B_n Tamari order is really a lattice. Before we can prove that, we need some preliminary results.

Let M_n denote the n-tuples with entries in $[0, n-1] \cup \{\infty\}$, with the Cartesian product order. Let $M_n^{(i)}$ denote the elements of M_n which satisfy condition (i) of Proposition 3. Let $M_n^{(ii)}$ denote the elements of M_n which satisfy condition (ii) of Proposition 3.

Proposition 5. *There exist maps* $\uparrow : M_n^{(ii)} \to T_n^B$, $\downarrow : M_n^{(i)} \to T_n^B$, *which satisfy the following conditions:*

(1) $$f \leq r(S) \text{ iff } \uparrow(f) \leq S$$

(2) $$r(S) \leq f \text{ iff } S \leq \downarrow(f).$$

Proof. Let $f \in M_n^{(ii)}$. Define $g \in M_n$ inductively, as follows:

$$g_i = \max_{j \leq f_i}(g_{i-j} + j).$$

By construction, g satisfies (i), and g satisfies (ii) since f does. Thus, g is a bracket vector. Let $\uparrow(f)$ be the corresponding triangulation. Now statement (1) is clear.

Let $f \in M_n^{(i)}$. Define $g \in M_n$, as follows: set $g_i = f_i$, unless $f_i \geq i$, and $f_{n+i-f_i} \neq \infty$. In this case, set g_i to be the largest number less than f_i such that $f_{n+i-g_i} = \infty$ or $g_i < i$. By construction, g satisfies (ii), and it is a straightforward check that g will also satisfy (i), since f does. (2) is also clear. $\square$

Using these maps, we can prove that meet and join exist in T_n^B by giving simple descriptions of them.

Proposition 6. *The Tamari order on* T_n^B *is a lattice. The lattice operations are as follows: For* $S, T \in T_n^B$, $S \vee T = \uparrow(\max(r(S), r(T)))$ *and* $S \wedge T = \downarrow(\min(r(S), r(T)))$.
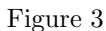
Proof. It is clear that, in M_n, the join of $r(S)$ and $r(T)$ is $\max(r(S), r(T))$. Now, since $\max(r(S), r(T)) \in M_n^{(ii)}$, for any $W \in T_n^B$, $W \geq S$ and $W \geq T$ iff $r(W) > \max(r(S), r(T))$ iff $W > \uparrow(\max(r(S), r(T)))$, so $\uparrow(\max(r(S), r(T))) = S \vee T$. The same argument holds for $S \wedge T$, once we observe that $\min(r(S), r(T)) \in M_n^{(i)}$. $\square$

This completes the proof of Theorem 1. The Hasse diagram of T_B^3 is shown in Figure 5, at the end of the paper. $\square$
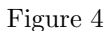
Noncrossing partitions

The A_n noncrossing partitions, NC_n^A, are partitions of $n+1$ into sets such that if $v_1, \ldots, v_{n+1}$ are $n+1$ points on a circle, labelled in cyclic order, and if $B_1, \ldots, B_r$ are the convex hulls of the sets of vertices corresponding to the blocks of the partition, then the B_i are non-intersecting.

There is a bijection from T_n^A to NC_n^A as follows. For $S \in T_n^A$, erase all the green chords and exterior edges of S, together with the vertices 0 and $n+2$. Then move the endpoints of each red chord ij a little bit, the lower end point a little clockwise, the higher endpoint a little counterclockwise (so i and j are both on the upper side of the chord). These chords now divide the vertices in $[n+1]$ into subsets, which form a noncrossing partition by construction. Figure 3 shows the triangulation from Figure 1, together with the noncrossing partition which it induces: $\{14, 23, 5\}$.

Figure 3

Note that the noncrossing partitions are often considered as being ordered by refinement; this order is quite different from the Tamari order.

As defined by Reiner [Rei], the B_n noncrossing partitions, NC_n^B, are partitions of the set $1, \ldots, n, \bar{1}, \ldots, \bar{n}$, which have the properties that the partition remains fixed under interchanging barred and unbarred elements, and that if $2n$ points are chosen around a circle and labelled cyclically $v_1, \ldots, v_n, v_{\bar{1}}, \ldots, v_{\bar{n}}$, then the convex hulls of the vertices corresponding to the blocks of the partition do not intersect.

We now define a map ψ from T_n^B to NC_n^B, analogous to that in type A. Erase all green chords and exterior edges. Move both endpoints of mixed red chords slightly counterclockwise. Move the endpoints of pure red chords slightly together (so that the vertices both lie on the side of the chord which includes the larger part of the polygon). Erase the vertices $n + 1$ and $\overline{n + 1}$. The remaining vertices are now partitioned by the red chords, in what is clearly a B_n noncrossing partition. Figure 4 shows the triangulation from Figure 2, together with the B_n noncrossing partition which it induces: $\{1\bar{2}\bar{5}\bar{6}, 34, \bar{1}256, \bar{3}\bar{4}\}$.

Figure 4

Proposition 7. *The map ψ is a bijection from T_n^B to NC_n^B.*

Proof. We remark first that T_n^B and NC_n^B have the same cardinality, $\binom{2n}{n}$ (see [Sim] and [Rei] respectively). Thus, it suffices to show that ψ is an injection.

Fix $S \in T_n^B$. Let T be a triangulation in the same fibre as S. Pick $i \in [n]$. Starting at $i - 1$, search counter-clockwise around the $(2n + 2)$-gon for the first vertex whose label is in the same block as i in $\psi(S)$. Let this vertex be v.

Consider first the case that $v = i-1$. Then there must be no red chords connected to i in T, so $C_i(T)$ is an edge segment. Next consider the case where $i - 1 > v \geq 1$. Then there must be a red chord iv in T. This chord cannot be $C_v(T)$, so it must be $C_i(T)$. Finally consider the case where $v = \bar{j}$. Then T must contain the red chord $i\left(\overline{j+1}\right)$, and it cannot contain any chord $i\bar{k}$ with $k < j + 1$. Thus $C_i(T)$ must be $i\left(\overline{j+1}\right)$. Similarly, if $v = n$, then $C_i(T)$ must be $i\bar{1}$. Finally, if v is unbarred and $n > v \geq i$, then there must be a red chord $(i - 1)(v + 1)$, so $C_i(T)$ is an edge segment.

Since the $C_i(T)$ suffice to determine T, it follows that S and T coincide, and ψ is an injection, as desired. $\square$

EL-SHELLABILITY

Recall that an element x of a lattice L is said to be left modular if, for all $y < z \in L$,

$$(3) \qquad\qquad (y \vee x) \wedge z = y \vee (x \wedge z).$$

In this section we shall prove the following theorem:

Theorem 2. *T_n^B has an unrefinable chain of left modular elements.*

The analogous fact that T_n^A posesses a unrefinable chain of left modular elements was first proved by Blass and Sagan [BS].

It was shown in [Liu] that a lattice having an unrefinable chain of left-modular elements has an EL-labelling. In particular, this shows that the order complex of any interval in such a lattice is shellable and hence contractible or homotopy equivalent to a wedge of spheres. For more on EL-labelling and EL-shellability, see [Bjö]. Neither T_n^A nor T_n^B are graded, and therefore the EL-shellings are not pure. For more on ungraded EL-shellability, see [BW1, BW2].

Thus, Theorem 2 implies the following corollary:

Corollary. *T_n^B is EL-shellable.*

Proof of Theorem 2. For $1 \leq i \leq n$ and $t \in [1, n - 1] \cup \{\infty\}$, let $S_{i,t}$ denote the triangulation with bracket vector as follows:

$$r_j(S_{i,t}) = \begin{cases} 0 & \text{for } j < i \\ t & \text{for } j = i \\ \infty & \text{for } j > i. \end{cases}$$

Lemma 9. *$S_{i,t} \in T_n^B$ is left modular.*

Proof. The proof is just a calculation, verifying (3). Let $Y < Z \in T_n^B$. Let $r(Y) = (y_1, \ldots, y_n)$, $r(Z) = (z_1, \ldots, z_n)$. For ease of notation, we split into three cases, when $t = \infty$, when $t \leq i - 2$, and when $i - 1 \leq t < n$.

Suppose first that $t = \infty$. We observe that

$$r(Z \wedge (S_{i,\infty} \vee Y)) = (y_1, \ldots, y_{i-1}, z_i, \ldots, z_n) = r((Z \wedge S_{i,\infty}) \vee Y),$$

as desired.

Suppose next that $t \le i - 2$. Applying Proposition 5, we see that:

$$r(S_{i,t} \vee Y) = (y_1, \ldots, y_{i-1}, \max_{0 \le j \le t} y_{i-j} + j, \infty, \ldots, \infty)$$

$$r(Z \wedge (S_{i,t} \vee Y)) = (y_1, \ldots, y_{i-1}, \min(z_i, \max_{0 \le j \le t}(y_{i-j} + j)), z_{i+1}, \ldots, z_n)$$

$$r(Z \wedge S_{i,t}) = (0, \ldots, 0, \min(z_i, t), z_{i+1}, \ldots, z_n)$$

$$r((Z \wedge S_{i,t}) \vee Y) = (y_1, \ldots, y_{i-1}, \max_{0 \le j \le \min(z_i, t)}(y_{i-j} + j), z_{i+1}, \ldots, z_n).$$

Thus, we need only check that

$$\min(z_i, \max_{0 \le j \le t}(y_{i-j} + j)) = \max_{0 \le j \le \min(z_i, t)}(y_{i-j} + j).$$

For $j \le z_i$, we have that $z_i \ge z_{i-j} + j \ge y_{i-j} + j$, so

$$\max_{0 \le j \le \min(z_i, t)}(y_{i-j} + j) \le z_i.$$

Thus, we alter nothing by rewriting:

$$\max_{0 \le j \le \min(z_i, t)}(y_{i-j} + j) = \min(z_i, \max_{0 \le j \le \min(z_i, t)}(y_{i-j} + j)).$$

If $j > z_i$, then $y_{i-j} + j > z_i$, so

$$\min(z_i, \max_{0 \le j \le \min(z_i, t)}(y_{i-j} + j)) = \min(z_i, \max_{0 \le j \le t}(y_{i-j} + j))$$

and we are done.

The case where $i - 1 \le t < n$ is similar. Let t' denote the largest integer less than t and greater than $i - 1$ such that $z_{n+i-t'} = \infty$. If there is none, set $t' = i - 1$. Then

$$r(S_{i,t} \vee Y) = (y_1, \ldots, y_i, \max(t, \max_{0 \le j < i}(y_{i-j} + j)), \infty, \ldots, \infty)$$

$$r(Z \wedge (S_{i,t} \vee Y)) = (y_1, \ldots, y_i, \min(z_i, \max(t', \max_{0 \le j < i}(y_{i-j} + j))), z_{i+1}, z_n)$$

where t has turned into t' because of the application of $\downarrow$.

On the other hand

$$r(Z \wedge S_{i,t}) = (0, \ldots, 0, \min(z_i, t'), z_{i+1}, \ldots, z_n)$$

$$r((Z \wedge S_{i,t}) \vee Y) = (y_1, \ldots, y_{i-1}, \max(\min(z_i, t'), \max_{0 \le j \le \min(i-1, z_j)}(y_{i-j} + j)),$$

$$z_{i+1}, \ldots, z_n).$$

The remainder of the argument is similar to the previous case. This completes the proof of Lemma 9. $\square$

Now observe that $\hat{0} \lessdot S_{n,1} \lessdot S_{n,2} \lessdot \cdots \lessdot S_{n,\infty} \lessdot S_{n-1,1} \lessdot \cdots \lessdot S_{1,\infty} = \hat{1}$ forms an unrefinable chain in T_n^B. This proves Theorem 2. $\square$

Homotopy types of intervals

As we have already remarked, the fact that T_n^B is EL-shellable implies that the order complex of any interval is either contractible or has the homotopy type of a wedge of spheres. In this section, we shall prove that it is in fact either contractible or homotopic to a single sphere. One reason that such a result is of interest is that it implies that the Möbius function of any interval in T_n^B is 0, -1, or 1.

Theorem 3. *The order complex of an interval in T_n^B is either contractible or homotopy equivalent to a single sphere.*

Proof. An element of a lattice called a join irreducible if it cannot be written as the join of two stricty smaller elements. We now prove some results concerning the join irreducibles of T_n^B.

For $1 \le t \le i-1$, let $W_{i,t}$ denote the triangulation whose bracket vector consists of t in the i-th place, all the other entries being zero.

For $i \le t < n$, let $W_{i,t}$ denote the triangulation defined by:

$$r_j(W_{i,t}) = \begin{cases} t & \text{for } j = i \\ \infty & \text{for } j = n + i - t \\ 0 & \text{otherwise} \end{cases}$$

Let $W_{i,\infty}$ denote the triangulation whose bracket vector consists of a single ∞ in the i-th place, all the other entries being zero.

Write $\mathcal{W}$ for the set of all the $W_{i,t}$.

Proposition 8. *The join irreducibles of T_n^B are exactly $\mathcal{W}$.*

Proof. It is easy to see that these elements are join irreducible and that any element of T_n^B can be written as the join of those $W_{i,t}$ below it.   □

For $S < T$, write $\mathcal{W}(S,T)$ for the set of join irreducibles of T_n^B which lie below T but not below S.

We now define the EL-labelling γ of [Liu]. Let L be a lattice, and let $\hat{0} = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_r = \hat{1}$ be an unrefinable chain of left modular elements. Let $\mathcal{W}_i$ be the set of join irreducibles below x_i but not below x_{i-1}. For $y < z$ in L, let $\mathcal{W}(y,z)$ be the set of irreducibles below z but not below y. For any $S \lessdot T$ in L, label the corresponding edge of the Hasse diagram by:

$$\gamma(S,T) = \min\{i \mid \mathcal{W}_i \cap \mathcal{W}(S,T) \neq \emptyset\}.$$

Proposition 9 [Liu]. *For L a lattice with an unrefinable left modular chain, the labelling γ defined above is an EL-labelling.*   □

We now interpret this labelling in our context. Observe that $W_{i,t}$ is that unique join irreducible that lies below $S_{i,t}$ but not below those $S_{j,p}$ below $S_{i,t}$. Instead of numbering the $S_{i,t}$, we proceed as follows. We put a new linear order, $\prec$, on $\mathcal{W}$, so that:

$$W_{n,1} \prec W_{n,2} \prec \cdots \prec W_{n,\infty} \prec W_{n-1,1} \prec \cdots \prec W_{1,\infty},$$

in other words, so that:

(4) $W_{i,t} \prec W_{j,p}$ iff $S_{i,t} < S_{j,p}$.

Now, we label the edges of the Hasse diagram of T_n^B by join irreducibles: if $S \lessdot T$, we label the edge (S,T) by the minimal element of $\mathcal{W}(S,T)$ (under $\prec$). Clearly, this is equivalent to the labelling defined by [Liu], and is therefore an EL-labelling.

Lemma 10. *If $Y \lessdot Z$, such that $r_i(Y) = r_i(Z)$ for $i \neq k$, then $\gamma(Y, Z) = W_{k,t}$ for some $1 \leq t \leq \infty$.*

Proof. Consider separately the cases where $r_k(Y) \leq k-2$, where $k-1 \leq r_k(Y) \leq \infty$, and where $r_k(Y) = \infty$. $\square$

Recall from [BW2] that given a poset with an EL-labelling, the order complex of an interval $[y, z]$ is homotopic to a wedge of spheres, one for each unrefinable chain from y to z such that the labels strictly decrease as one reads up the chain. Such chains are called *decreasing chains*.

Thus, Theorem 3 will follow from the following lemma:

Lemma 11. *For $Y < Z \in T_n^B$, there is at most one decreasing chain from Y to Z.*

Proof. Let $Y = T_0 \lessdot T_1 \lessdot \cdots \lessdot T_r = Z$ be a decreasing chain from Y to Z.

For each i, let k_i denote the unique place (provided by Lemma 6) where $r(T_i)$ and $r(T_{i+1})$ differ. By Lemma 11, the label on the edge (T_i, T_{i+1}) is W_{k_i, t_i} for some t_i. Since the labels are decreasing by assumption, k_0 must be the first index where $r(Y)$ and $r(Z)$ differ. Thus $r(T_1)$ must be the smallest legal bracket vector obtainable by increasing the k_0 position of $r(T_0)$. This determines T_1 uniquely, and the remaining T_i are determined inductively, proving the lemma. $\square$

This completes the proof of Theorem 3. $\square$

An exact description of when the homotopy type of the interval $[Y, Z]$ in T_n^B is homotopic to a sphere, and when it is contractible (expressed in terms of the bracket vectors of Y and Z), has been worked out in [Sa2].

Generalizing to Type BD_n^S

Here we fix n and a subset S of $[n]$. We will be operating in type BD_n^S, a notation introduced in [Rei] which we now explain. This is not a type in the usual sense. Rather, it refers to a certain hyperplane arrangement between those associated to B_n and D_n.

Recall that a root system gives rise to a hyperplane arrangement by taking all the hyperplanes through the origin perpendicular to roots. The B_n arrangement therefore consists of all those hyperplanes defined by $x_i \pm x_j = 0$, together with those defined by $x_i = 0$, for $1 \leq i, j \leq n$, while the D_n arrangement consists only of those hyperplanes defined by $x_i \pm x_j = 0$ for $1 \leq i, j \leq n$. Now, for $S \subset [n]$, the BD_n^S hyperplane arrangement consists of those hyperplanes defined by $x_i \pm x_j = 0$ together with $x_i = 0$ for $i \notin S$. When $S = \emptyset$ we recover the B_n arrangement, while if $S = [n]$ we obtain the D_n arrangement.

The B_n partitions, Π_n^B, are by definition those partitions of the set $\{1, \ldots, n, \bar{1}, \ldots, \bar{n}\}$ which are fixed under the map interchanging i and $\bar{i}$, and such that there is at most one block which contains any i and $\bar{i}$ simultaneously. This is a suitable definition of Π_n^B because its elements are naturally in bijection with the elements of the intersection lattice of the B_n arrangement. NC_n^B is a subset of Π_n^B.

The intersection lattice of the BD_n^S hyperplane arrangement is a subset of that of type B_n. This allows a natural definition of BD_n^S partitions, Π_n^S, as a subset of Π_n^B. By this approach, one obtains that Π_n^S consists of those partitions of Π_n^B which do not contain any block consisting solely of $\{i, \bar{i}\}$ for $i \in S$. In [Rei], Reiner defined NC_n^S, the noncrossing partitions of type BD_n^S, by $NC_n^S = NC_n^B \cap \Pi_n^S$.

In particular, Reiner defined the noncrossing partitions of type D_n to be the noncrossing partitions of type $BD_n^{[n]}$. Since [Rei] was written, it has become clear that there is a more natural definition of noncrossing partitions of type D, see Athanasiadis and Reiner [AR]. Thus, even though type $BD_n^{[n]}$ is associated to the D_n hyperplane arrangement, it should not be confused with type D_n.

Let T_n^S be those triangulations which correspond under ψ to partitions in NC_n^S. We can describe them more directly as follows:

Lemma 12. T_n^S *consists of those triangulations which do not contain the triangles* $i, i+1, \overline{i+1}$ *and* $\bar{i}, i+1, \overline{i+1}$ *for any* $i \in S$. T_n^S *can also be characterized as the set of triangulations* T *such that* $r_i(T) \neq n-1$ *for any* $i \in S$.

Proof. We prove the first statement by showing that $\psi(T)$ contains the block $\{i, \bar{i}\}$ iff T contains the triangles $i, i+1, \overline{i+1}$ and $\bar{i}, i+1, \overline{i+1}$.

If $\psi(T)$ contains the block $\{i, \bar{i}\}$, then T contains red chords $\bar{i}(i+1)$ and $i\left(\overline{i+1}\right)$. Therefore, T must also contain either $i\bar{i}$ or $(i+1)\left(\overline{i+1}\right)$. The first of these chords would be red, which contradicts the presence of $\{i, \bar{i}\}$ in $\psi(T)$. So T must contain also contain the edge $(i+1)\left(\overline{i+1}\right)$, so it contains the two desired triangles.

Conversely, suppose T contains the two desired triangles. It is immediate that $(i+1)\left(\overline{i+1}\right)$ will be green while $i\left(\overline{i+1}\right)$ and $\bar{i}(i+1)$ will be red, and therefore $\psi(T)$ contains the block $\{i, \bar{i}\}$. This completes the proof of the first statement.

The second statement follows immediately from the first. $\square$

The remainder of the paper is devoted to the proof of the following theorem, which generalizes Theorems 1, 2, and 3 to the broader context of type BD_n^S.

Theorem 4. T_n^S *admits a lattice structure which is a quotient of that on* T_n^B. T_n^S *posesses an unrefinable chain of left modular elements, which implies that it is EL-shellable. Further, the order complex of any interval is either contractible or homotopic to a single sphere.*

Proof. We define an equivalence relation $\sim_S$ on T_n^B as follows: two non-identical triangulations are equivalent iff they differ in that one of them, say T, is not in T_n^S, and the other is the triangulation obtained by removing the diameter of T and replacing it with the other possible diameter. It is immediate from Lemma 12 that this triangulation will be in T_n^S.

We can express equivalence in terms of bracket vectors by saying that V and W are equivalent if there is some $k \in S$ such that $r_j(V) = r_j(W)$ for all $j \neq k$, and $r_k(V) = n-1$ while $r_k(W) = \infty$.

An equivalence relation $\sim$ on a lattice L is said to be a congruence relation if the lattice operations pass to equivalence classes. In this case, there is an induced lattice structure on the equivalence classes (see [Grä]).

Lemma 13. *The relation* $\sim_S$ *on* T_n^B *is a congruence relation.*

Proof. It suffices to show that for $V \sim_S W \in T_n^B$ and $Z \in T_n^B$, $V \vee Z \sim_S W \vee Z$ and $V \wedge Z \sim_S W \wedge Z$. But these are both clear from the descriptions of the lattice operations in T_n^B in terms of bracket vectors. $\square$

Since the equivalence classes of $\sim_S$ each contain a single element of T_n^S, the induced lattice structure on $T_n^B / \sim_S$ gives rise to a lattice structure on T_n^S. Let us write $<_S$ for the order induced in this way on T_n^S.

There is another way to induce a poset structure on T_n^S, namely that induced by its inclusion in T_n^B. Let us write $<_B$ for the poset structure induced on T_n^S in this way. As we now prove, $<_B$ and $<_S$ coincide (and so, once we have proved the lemma, we will drop the subscripts).

Lemma 14. *The two poset structures on T_n^S, $<_B$ and $<_S$, coincide.*

Proof. Observe that T_n^S is closed under the meet in T_n^B. Thus, $(T_n^S, <_B)$ has a meet, namely the meet in T_n^B restricted to T_n^S. But this coincides with the meet defined for $(T_n^S, <_S)$. Knowing that meets exist in both poset structures, and that they coincide, implies that the two structures themselves coincide. $\square$

We shall generally prefer to consider the poset structure on T_n^S as being induced by $\sim_S$. Note that T_n^S does not form a sublattice of T_n^B, because T_n^S is not closed under the join in T_n^B.

It is immediate that the property of being left modular passes to equivalence classes, so T_n^S has a unrefinable chain of left modular elements, and is therefore EL-shellable. This unrefinable chain is shorter than that of T_n^B, because $S_{i,n-1} \sim_S S_{i,\infty}$ for $i \in S$.

It is easy to see that the join irreducibles of T_n^S are those $W_{i,t}$ such that either $i \notin S$ or $t \neq n-1$; again, they are in bijection with the elements of the left modular chain. Write $\mathcal{W}^S$ for the set of all the join irreducibles of T_n^S. As before, we define an order $\prec$ on $\mathcal{W}^S$ so that (4) is satisfied. In fact, this is just the order induced on $\mathcal{W}^S$ from its inclusion in $(\mathcal{W}, \prec)$.

Lemma 15. *If $Y \lessdot Z$ in T_n^S, then either $Y \lessdot Z$ in T_n^B or there exists some Z' such that $Y \lessdot Z' \lessdot Z$ in T_n^B, and $Z' \sim_S Z$. In the latter case, the bracket vectors of Y, Z', and Z all coincide except in one place, where Z' has $n-1$ and Z has ∞.*

Proof. Suppose that $Y \lessdot Z$ in T_n^S but $Y \not\lessdot Z$ in T_n^B. So there exists some Z' in T_n^B, $Y < Z' < Z$. When we pass to equivalence classes with respect to $\sim_S$, one of the strict inequalities becomes an equality, so either $Z' \sim_S Y$ or $Z' \sim_S Z$. The first case is impossible, because an element of T_n^S is the top element of its equivalence class mod $\sim_S$. Thus, the second case holds, which proves that $Z' \sim_S Z$, and therefore that $Z' \lessdot Z$, and, since we have shown that the only element in T_n^B strictly between Y and Z is Z', it follows that $Y \lessdot Z' \lessdot Z$.

The bracket vectors of Z' and Z differ in one place, say k, and clearly $r_k(Z') = n-1$ and $r_k(Z) = \infty$. The bracket vector of Y differs from that of Z' in one place, and since $Y \in T_n^S$, $r(Y)_k \neq n-1$, so Y differs from Z' in the k-th place. $\square$

For $Y < Z \in T_n^S$, write $\mathcal{W}^S(Y, Z)$ for the set of join irreducibles of T_n^S below Z but not below Y. Now, for $Y \lessdot Z$ in T_n^S, we label the edge (Y, Z) by the first element (with respect to $\prec$) of $\mathcal{W}^S(Y, Z)$. By the result of Liu already cited, this is an EL-labelling.

Using Lemma 15, we see that Lemma 10 holds in T_n^S; the same proof goes through. The argument used to prove Lemma 11 now goes through to show that there can be at most one decreasing chain from Y to Z in T_n^S. The final statement of Theorem 4 follows. $\square$

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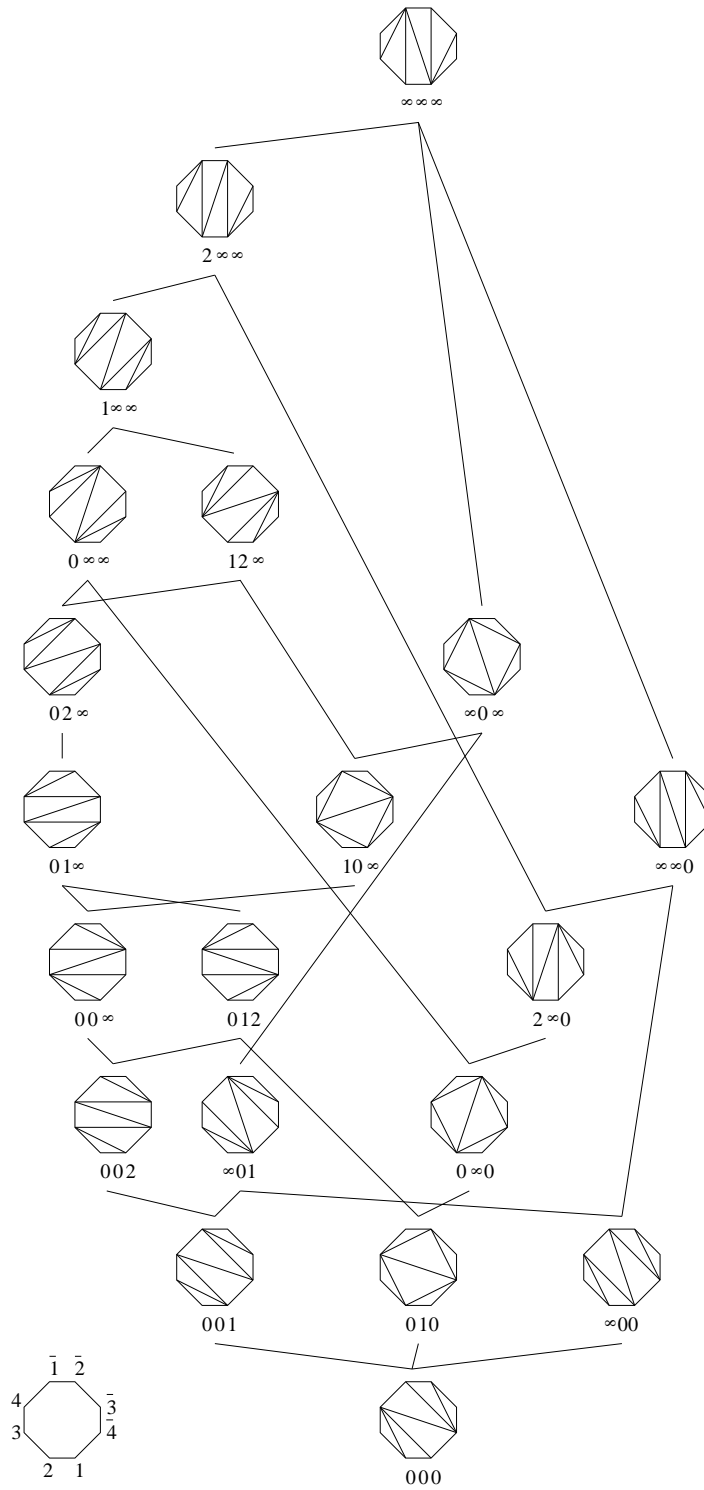[Tam]   D. Tamari, *The algebra of bracketings and their enumeration*, Nieuw. Arch. Wisk. **10** (1962), 131–146.

Figure 5